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SILVER SPRING, MARYLAND

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ANALYTIC EXPRESSIONS FOR THE PROBABILITY OF PENETRATING A POINT DEFENSE

by

Girard L. Ordway and
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1 November 1958

Work sponsored under Subcontract No. 77745 by the Applied Physics Laboratory, The John Hopkins University, operating under Contract NOrd 7386 with the Bureau of Naval Weapons, Department of the Navy.

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ABSTRACT

A method is developed for computing the probability that an air attack will penetrate a point defense made by a surface-to-air missile system. Analytic expressions are obtained from a model for the probability of penetration by at least one of k attackers, as a function of k and of the spacing between successive attackers, the number of shots that can be made at an approaching attacker, and the single-shot kill probability of the defensive missiles. The model is initially formed in terms of a constant spacing between attackers, but modifications are found that permit generalization to obtain results for any spacing between attackers. The mean number of attackers required for penetration can be approximated very closely by a formula that is valid throughout the range of tactical interest. The number of attackers required to saturate the defensive capabilities and thus to ensure penetration is shown to be only slightly greater than the mean.

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I. INTRODUCTION

1.1 The problem of determining the capability of a surface-to-air defensive system to withstand an air attack is extremely complicated in the general case. Solutions can easily be obtained if the attackers arrive at the defense perimeter either widely spaced or simultaneously, but in general the defense is faced with a choice of firing doctrine and with dealing with a large number of compound events.

1.2 What one expects of a mathematical model of a situation is a formulation that allows the easy calculation of results and also permits a study of the sensitivity of the results to a variation in the parameters of the problem. Owing to the complexity of the problem many previous studies of the air defense problem have resorted to gaming and simulation of various possible engagements to determine the penetration probabilities of attacking forces. This is a cumbersome method and requires the repetition of a large number of simulated engagements on an electronic computer in order to obtain valid results. If any change is made in the performance characteristics of the defense or attack missiles, another lengthy series of calculations must be made.

1.3 To avoid these difficulties an analytical model has been developed in which the basic parameters were deliberately chosen to be abstract quantities not intended to be defined numerically until a specific application was required. This method of formulation has a number of advantages. The physical identification of the parameter can be changed to suit the needs of a particular kind of defensive situation, so that it is possible to consider within the framework of the model defensive systems that are limited in effectiveness by the range of the missiles or by the effectiveness of multilauncher defensive systems with different degrees of coordination of fire. In addition to its versatility in dealing with a variety of physical situations, the model provides a simple and rapid means of calculating results. The probabilities that the attackers penetrate the defenses can be computed for many cases on a desk calculator, and the effects of changing parameters can be investigated by this means. If it is desired to obtain the results in fine detail it will still be necessary to use an electronic computer; but the results obtained are in closed form and do not depend on gaming, so that they can be obtained quickly and easily.

II. RIGOROUS FORM OF THE PENETRATION PROBABILITIES

BASIC PARAMETERS IN THE MATHEMATICAL MODEL

2.1 The parameters required to describe an attack situation are the number of shots that can be made at a single attacker, the spacing between the attackers, the single-shot kill probability of the defensive missiles, and the number of attackers.

2.2 The number of shots that it is possible to take at a single attacker is called $\underline{a}$. The physical identification of this parameter will depend on the range at which the attacker can be intercepted, which in turn depends on the type of attack and the range and altitude the defensive missile can reach. The physical identification will also depend on the rate of fire of the defensive missiles and the speed of the attackers.

2.3 The spacing between attackers is expressed in terms of the fraction of the interval between successive shots by the defense, (s/τ). This dimensionless choice of time unit facilitates the counting of obligations assumed by the defense, which are imposed at the appropriate spacings of a shot. It can be seen immediately that if the attackers arrive at less than unit spacing, the defense will ultimately be overwhelmed. The physical identification of the spacing expressed in these dimensionless units depends on the actual time spacing of the attackers, s , and on the time between successive firings of the defensive missiles, τ .

2.4 Further parameters needed in setting up the defense model are the single-shot kill probability of the defensive missiles, p , and the number of attackers, k . Both of these quantities have direct physical significance. The complementary probability ($q = 1 - p$) of a single-shot failure is also useful.

2.5 The parameter a , the number of shots that it is possible to take at a single approaching attacker, is the most difficult to evaluate because it depends on the tactics and physical characteristics of both the attack and the defense. The parameter obviously depends on the range of the attacker at the time when effective defensive fire can commence; however, this may be limited by the capability of the defense to detect and evaluate the attack threat. The number of missiles that can be fired as a single attacker crosses the defended zone is given by

$$a = 1 + \frac{(R_a - R_o)}{\tau V_a} + \frac{R_o}{\tau V_d} \, ; \tag{2.1}$$

where

R_a = effective range of the defensive missile, or the range of attack detection and evaluation, whichever is smaller (outer perimeter of defense zone);

R_o = range at which an attack missile can damage the defended target (inner perimeter of defense zone);

V_a = velocity of the attacking missiles;

V_d = velocity of the defensive missiles.

ASSUMED DISTRIBUTION OF DEFENSIVE FIRE

2.6 The probability that the defenses will be penetrated by an attack is a function of a , s/τ, p, and k ; but the way in which these quantities are combined to obtain the penetration probabilities of the attackers depends on an assumed doctrine of fire for the defense. The model that has been developed specifies that the defense shall direct its fire at the nearest attacker until it is hit. In order to apply this doctrine in practice it may be necessary to reassign a missile already in flight to a new target as soon as it becomes known that a kill has been scored. This would require very close coordination of the launching and the guidance systems, and represents a clear limitation on the applicability of the model. However, the model that has been developed would also apply to the case where the flight reliability and the lethality of the defensive missiles are so high that the missile effectiveness is limited by events in the launch and boost phases, which would permit an evaluation to be made soon after launching.

INTEGRAL SPACING OF ATTACKERS

2.7 Using this firing doctrine, it is possible to build up the probability of firing at a given attacker in any of the time intervals of duration τ each during which it is in range. The first case considered is for attackers arriving spaced an integral number of shots apart (s/τ is integral).

2.8 Consider a firing battery that has an opportunity to fire $\underline{a}$ shots at an approaching attacker, and assume that an attacking file of uniformly spaced planes is approaching at a rate such that the battery can fire $\underline{s}$ shots at the first plane in the file before the next plane gets within range. The doctrine will be to fire at a plane until it is hit and then to fire at the next plane.

2.9 The operation of this doctrine is shown schematically in Figure 1 for the case where the battery has time for $a = 6$ shots at an approaching plane, and the attackers are spaced $s = 3$ shots apart. Several possible sequences of events are shown. The lower route shows the case of three failures (marked by F) in firing at the first plane, followed by a success (marked by S) on the fourth shot. After this it is possible to fire at the second plane, which has been within range for a length of time during which one shot could have been fired if there had been a battery available. A continuation along the bottom route in the diagram shows a failure in shooting at the second plane, followed by a success which makes it possible to fire at the third plane as it comes into range. Another possible sequence of events in shown in the upper route. Here there is a failure followed by a success in firing at the first plane. The battery is then idle for a one-shot spacing until the second plane comes into range. In the sequence shown the battery scores a success against the second plane on the first shot at it and then is idle for a two-shot spacing until the third plane comes into range.

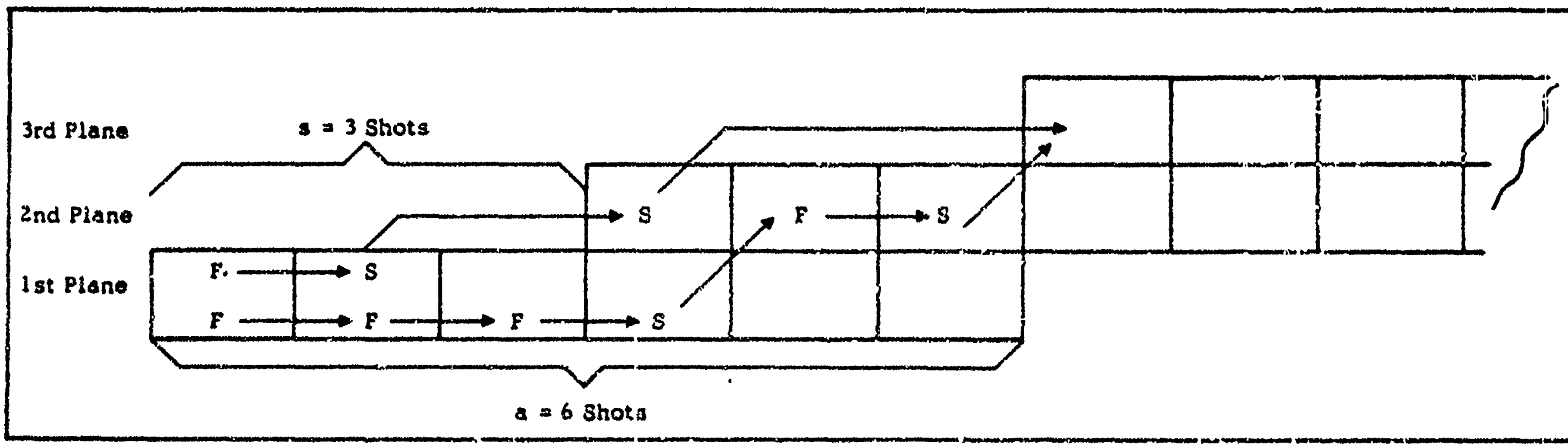

FIGURE 1. POSSIBLE FIRING SEQUENCES

2.10 Matrix Formulation. It is possible to keep track of these various possible courses of events by setting up a matrix that transforms the probability of shooting at a given plane to the probability of shooting at the next plane in the file. The first case considered will be that of uniformly spaced planes approaching an integral number of shots apart. The probabilities, $b_j^{(k)}$, of firing at the k-th plane during each of the a periods that it is in range can be arrayed in a row matrix (of a elements)

$$B_k = (b_1^{(k)} b_2^{(k)} b_3^{(k)} \ , \ . \ . \ b_a^{(k)}) \ . \qquad (2.2)$$

The $b_j^{(k)}$ are functions of the single-shot kill probabilities. If the probability of making a kill with a single shot is p and the probability of failing to make a kill is $[(1-p) = q]$, then the probability, Q_k, that the k-th plane penetrates the defense is equal to the probability of firing at it on the a-th shot and missing, i.e.,

$$Q_k = b_a^{(k)} q \ . \qquad (2.3)$$

TRANSFORMATION MATRIX

2.11 The elements $b_j^{(k)}$ of B_k can be obtained by successive matrix multiplication, as will now be shown. The first matrix, B_1, of shots at the first plane can be obtained by inspection. The event of successive firings at the first plane can occur as a result of repeated failures, and the probability of firing at the first plane on the j-th shot can be represented as the j-th element of a row matrix.

$$B_1 = (1 \ q \ q^2 \ . \ . \ . \ q^{a-1}) \ . \qquad (2.4)$$

2.12 The probability that the first plane penetrates, Q_2, is then q^a. It is now necessary to find a matrix that will multiply B_1 on the right to transform B_1 into B_2. For this purpose an a by a square matrix M is written with the element M_{ij} (in the i-th row and j-th column) chosen so as to transform the j-th element of B_1 into a contribution to the i-th element of B_2. The matrix M is given by

$$
\begin{array}{c}
\left(\begin{array}{cccccccccccc}
p & pq & pq^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & pq^{a-1} \\
p & pq & pq^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & pq^{a-1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
p & pq & pq^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & pq^{a-1} \\
0 & p & pq & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & pq^{a-2} \\
0 & 0 & p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & pq^{a-3} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & \cdot & p & pq & pq^2 & \cdot & \cdot & pq^{s-1} & pq^s \\
0 & 0 & 0 & \cdot & \cdot & 0 & 1 & q & \cdot & \cdot & q^{s-2} & q^{s-1}
\end{array}\right)
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} s \text{ rows}
\\
\underbrace{\hspace{6cm}}_{s \text{ columns}}
\end{array}
\qquad (2.5)
$$

The first column has $\underline{s}$ p's in it because a success in any of the first $\underline{s}$ shots at the first plane permits a shot at the next plane as soon as it comes into range. The rest of each of the first $\underline{s}$ rows is filled in by multiplying successive elements across the rows by q ; this occurs as a result of applying the firing doctrine, which specifies that after a miss one fires again at the same plane, which will have advanced one spacing. From the first element of the s-th row of the matrix a diagonal of p's extends downwards to the last row. Consulting the diagram of Figure 1 will make it clear that a success on the (s + j)th shot at the first plane will make it possible to fire at the second plane on the (j + 1)st shot. Again the rows are filled in by multiplying successively across by q . The first non-zero element in the last row is not p but unity because of the following interpretation of the firing doctrine. The first plane is only in range for $\underline{a}$ shots, and it would be impossible to fire (a + 1) shots at it. After the a-th shot at the first plane, therefore, the missile launcher is directed more or less in desperation at the second plane. The contingency that the launcher may not be there to direct at anything is taken into account by computing the penetration probability of the first plane. The last row of the matrix is filled in like the others by multiplying successively across by q.

2.13 The distribution of probabilities of shots at the second plane can now be obtained as

$$B_2 = B_1 M \ , \tag{2.6}$$

and the penetration probability of the second plane

$$Q_2 = B_1 M \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ q \end{pmatrix} \tag{2.7}$$

where the column matrix on the right has a rows and so multiplies the probability of making an a-th shot by the probability of a single-shot failure.

2.14 On substituting for B_1 and M their values given above, one finds after making reductions based on the relation $p + q = 1$,

$$Q_2 = q^a[1 + (a - s) pq^{s-1}] \quad . \tag{2.8}$$

2.15 Multiplication to Obtain Successive Probabilities. The matrix M was derived so as to transform the array of probabilities of shots at a given plane to the array for the next plane. It is thus possible to find B_k and P_k for the k-th plane by multiplying B_1 by successive powers of M .

$$B_k = B_1 M^{k-1} \tag{2.9}$$

and

$$Q_k = B_1 M^{k-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ q \end{pmatrix} \text{(a rows)} \quad . \tag{2.10}$$

2.16 Matrix Equation for Penetration Probabilities. The individual penetration probabilities, Q_k, can be used to obtain the probability P_k that at least one out of an approaching file of k planes penetrates the defense. This is given by

$$P_k = 1 - \prod_{n=1}^{k} (1-Q_n) \quad . \tag{2.11}$$

2.17 Equivalent Formulations. It is possible to transform the matrix M to other forms, mathematically equivalent, which may either result in a simplification of the computation or assist in setting up transformation matrices for more general cases than that of integral spacing.

2.18 As a preliminary it can be seen that if a matrix N is related to M by a similarity transformation R, i.e.,

$$N = R^{-1} MR \ , \tag{2.12}$$

then

$$Q_k = B_1 M^{k-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ q \end{pmatrix} = B_1 R (R^{-1} MR)^{k-1} R^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ q \end{pmatrix} \tag{2.13}$$

$$= B_1 RN^{k-1}R^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ q \end{pmatrix} \quad . \tag{2.14}$$

2.19 Several transformations, which can readily be generalized, are illustrated for the case where s = 1 and a = 3. Then setting up the transformation matrix as in the expression (2.5)

$$M = \begin{pmatrix} p & pq & pq^2 \\ 0 & p & pq \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{2.15}$$

2.20 A transformation is now shown that reduces the dimensionality of M from a̲ to (a - s) and permits the Q_k to be expressed in the form of a power series. For this purpose, let

$$R = \begin{pmatrix} 1 & p & p \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{2.16}$$

so that

$$R^{-1} = \begin{pmatrix} 1 & -p & -pq \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{2.17}$$

Then

$$N = R^{-1}MR = \begin{pmatrix} p & pq & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.18}$$

and

$$Q_k = (1\ \ 1\ \ 1)\,N^{k-1} \begin{pmatrix} -pq^2 \\ -pq \\ q \end{pmatrix} . \tag{2.19}$$

Now

$$(1\ \ 1\ \ 1)\,N = [\,(1-q)\,(1-q^2)\ \ 1\ ]$$

$$= (1\ \ 1\ \ 1) + (-q\ \ -q^2\ \ 0) \tag{2.20}$$

so that

$$Q_k = Q_{k-1} + (-q\ \ -q^2\ \ 0)\,N^{k-2} \begin{pmatrix} -pq^2 \\ -pq \\ q \end{pmatrix} . \tag{2.21}$$

From the position of the vanishing elements in the transformed matrices, it is apparent that the last term on the right can be expressed as a product of matrices of reduced order, i.e.,

$$(-q \;\; -q^2 \;\; 0)\, N^{k-2} \begin{pmatrix} -pq^2 \\ -pq \\ q \end{pmatrix} = (-q \;\; -q^2) \begin{pmatrix} p & pq \\ 0 & p \end{pmatrix}^{k-2} \begin{pmatrix} -pq^2 \\ -pq \end{pmatrix}$$

$$= pq^2 \;(1 \;\; q) \begin{pmatrix} p & pq \\ 0 & p \end{pmatrix}^{k-2} \begin{pmatrix} q \\ 1 \end{pmatrix}$$

$$= p^{k-1}\, q^3 \;(1 \;\; 1) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad . \qquad (2.22)$$

2.21 If this transformation is generalized, it is found that the Q_k can be expressed as a power series in pq^{s-1} with coefficients derived from a matrix of order $(a-s)$. Specifically,

$$Q_k = q^a \left[1 + \sum_{n=2}^{k} A_n \,(pq^{s-1})^{n-1} \right] \qquad (2.23)$$

where

$$A_n = (1 \;\; 1 \;\; 1 \;\; . \;\; . \;\; . \;\; 1)\, M*^{n-2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \text{(a-s elements)} \qquad (2.24)$$

and $M*$ is a square matrix with $(a-s)$ rows and columns (i and $j \leq a-s$) with elements equal to either zero or one.

$$M^*_{ij} = 1 \qquad (j > i-s) \qquad (2.25)$$

$$= 0 \qquad (j \leq i-s) \; . \qquad (2.26)$$

Furthermore if $\underline{s}$ is equal to or greater than half of $\underline{a}$, the matrix M^* has nothing but unit elements, and the matrix product reduces to

$$A_n = (a-s)^n \; . \qquad (2.27)$$

Thus the factor multiplying q^a in Q_k is part of a geometric series with a common ratio $(a-s)\,pq^{s-1}$. In some cases the work of computation may thus be considerably reduced.

TRANSFORMATION TO DISTRIBUTION OF FIRST SHOTS

2.22 The matrix M can be subjected to another similarity transformation, which results in the distribution of first shots at a given plane rather than the distribution of total probabilities of shooting at a given plane. Such a formulation may result in greater convenience of computation, and it also suggests a way of setting up the transformation matrix more easily in the general case of non-integral spacing.

2.23 A transformation is now shown that gives the distribution of first shots for the case of a=3 and s=1 . Again the method is readily generalized. Let

$$U = \begin{pmatrix} 1 & -q & 0 \\ 0 & 1 & -q \\ 0 & 0 & 1 \end{pmatrix} \qquad (2.28)$$

so that

$$U^{-1} = \begin{pmatrix} 1 & q & q \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \; . \qquad (2.29)$$

Then

$$Q_k = (1 \;\; q \;\; q^2)\, M^{k-1} \begin{pmatrix} 0 \\ 0 \\ q \end{pmatrix}$$

$$= (1 \;\; 0 \;\; 0)\, V^{k-1} \begin{pmatrix} q^3 \\ q^2 \\ q \end{pmatrix} \tag{2.30}$$

where

$$(1 \;\; 0 \;\; 0) = C_1 = B_1\, U \tag{2.31}$$

and

$$V = U^{-1} M U = \begin{pmatrix} p & pq & q^2 \\ 0 & p & q \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.32}$$

2.24 Inspection of a diagram for this case shows that the matrix V transforms an array of probabilities of taking a shot for the first time at a given plane into a corresponding array for the succeeding plane. The new column matrix on the right in the expression for Q_k multiplies the first-shot probabilities by the probabilities of the requisite number of failures to permit a penetration. The row matrix on the left is made up of a elements starting with a unit element, which is followed by a series of zeros. This expresses the fact that the sequence of events is initiated by firing for the first time at the first plane.

AVERAGE NUMBER OF SHOTS REQUIRED

2.25 If the formulation involving the distribution of total shots is used for the calculation, then the average number of shots fired at the k-th plane, N_k , is just the sum of the elements of B_k , which are the probabilities of firing at the k-th plane in the various subintervals of the time during which it is in range.

$$N_k = B_k \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix} (\underline{a} \text{ rows}) \quad . \tag{2.33}$$

2.26 If the distribution of first shots is used to formulate the problem, then the average number of shots at the k-th plane is obtained by multiplying the row matrix giving the distribution of first shots at the k-th plane, which equals

$$C_k = (1 \ 0 \ 0 \ . \ . \ . \ 0)V^{k-1} \quad , \tag{2.34}$$

on the right by a column matrix D , whose j-th element is

$$d_j = \sum_{n=0}^{a-j} q^n \quad . \tag{2.35}$$

2.27 The matrix D both converts the distribution of first shots back to total shots and sums the probabilities to get the average number of shots fired at a given plane. The number of shots fired at the individual attackers can be added together to get the total number of shots fired in an engagement.

SPACING OF ATTACKERS NOT A MULTIPLE OF THE FIRING INTERVAL

2.28 It is now possible to extend the matrix method of determining penetration probabilities to cases where the arrival spacing of the attackers, $\underline{s}$, is not an integral multiple of the firing interval, τ . The same general methods of procedure apply that were used in developing a solution for integral spacing, although some minor modifications are necessary. The firing interval τ is still assumed to be constant.

2.29 For the general case it is probably easier to set up the V matrix, which transforms the distribution of first shots. The procedure for finding the V matrix is now outlined. The principal modification required is a further subdivision of the attack time so as to obtain time units commensurable with the arrival and firing spacings. The time during which an attacker is in range is divided into $a\tau$ time intervals, and the probabilities of firing for the first time at the k-th plane in each of these intervals are displayed as the elements of a row matrix, $C^{(k)}$. The element v_{ij} of V transforms the probability of firing at the k-th plane in the i-th position into a contribution to the probability of firing at the $(k+1)^{st}$ plane in the j-th position, so that

$$c^{(k+1)} = \sum_{i=1}^{a\tau} c_i^{(k)} v_{ij} , \tag{2.36}$$

or in matrix notation

$$C^{(k+1)} = C^{(k)} V . \tag{2.37}$$

Now the shots at the $(k+1)^{st}$ plane are displaced $\underline{s}$ intervals with respect to those of the k-th plane, and a first shot can be fired at the $(k+1)^{st}$ plane only if a shot has been fired at the k-th plane τ intervals (or some integral multiple of τ intervals) previously. Thus, in general the only non-vanishing v_{ij} arise when

$$i = j + s - (n+1)\tau \; (n=0,\ 1,\ 2,\ .\ .\ .\ ) \tag{2.38}$$

and the value of the non-vanishing matrix element is

$$v_{ij} = pq^n . \tag{2.39}$$

2.30 There are two regions in which elements of V are formed exceptionally. If the spacing $\underline{s}$ is equal to or greater than 2τ some of the elements v_{i1} in the first column of V will be made up of a sum of terms. This occurs because there is time for several shots at the k-th attacker before the $(k+1)^{st}$ attacker comes into range, and thus a first shot at the $(k+1)^{st}$ attacker in the first position can result from several courses of events in firing at the k-th attacker. There is a probability p of hitting the first attacker immediately; if this happens the defense need not fire again until the second plane comes into range. On the other hand, if the first shot is a failure and the second a success there will still be time to fire at the second attacker as it comes into range; the probability of this event is pq. The matrix element v_{il} is formed from a sum of these terms. Non-vanishing terms occur when

$$i \leq l + s - \tau \tag{2.40}$$

and the value of v_{il} is

$$v_{il} = \sum_{n=0}^{n \leq (l+s-\tau-i)/\tau} pq^n (n=0, 1, 2, \ldots). \tag{2.41}$$

2.31 Although for purposes of mathematical convenience the duration of the engagement with a single attacker is considered to be divided into $a\tau$ sub-intervals, it is impossible for the defense to fire at the attacker after $(a-1)\tau + 1$ sub-intervals; i.e., there are $(\tau - 1)$ intervals in which it is physically impossible to fire at an attacker, and the existence of these intervals gives rise to the other exceptional kinds of matrix element.

2.31 Since it is impossible to fire at the k-th plane in the last $(\tau - 1)$ spaces, the doctrine states as a counsel of desperation that the $(k+1)^{st}$ attacker will be fired at in positions corresponding to the last $(\tau - 1)$ positions for the k-th attacker, i.e., for j given by

$$(a-1)\tau - s + 1 < j \leq a\tau - s + 1 \tag{2.42}$$

there will be non-vanishing matrix elements arising from positions for the k-th attacker given by

$$i = j + s - (n+1)\tau \qquad (n=0,1,2, \ldots) \tag{2.43}$$

and these non-vanishing elements are given by

$$v_{ij} = q^n . \tag{2.44}$$

Furthermore, for the unfortunate case where τ is greater than $\underline{s}$, new attackers are arriving faster than the defense can fire; and the defense will certainly be penetrated in time. It may be that the attackers press in so close together that the doctrine would require firing even at the $(k+1)^{st}$ plane in a physically impossible position. To account for this contingency, a matrix element is included that transforms the probability of firing at the k-th plane to a position for a virtual firing at the $(k+1)^{st}$ plane. It will still be impossible to fire at the $(k+1)^{st}$ plane, but the probability of firing will by now have been transferred to a position such that a further transformation will move it to a possible position for firing at the $(k+2)^{nd}$ or some later plane. This process can be accomplished by introducing a matrix element equal to unity in the physically impossible range, so that for

$$(a-1)\tau - s + 1 < a\tau - s + 1 \tag{2.45}$$

and

$$i = j + s \; , \tag{2.46}$$

$$v_{ij} = 1 \; . \tag{2.47}$$

2.33 The introduction of these virtual firings assumes that shots will be accounted for as long as there is any possibility at all of firing. Eventually there comes a time when the only firing that can be made is in a physically excluded region, in which case penetration certainly occurs. The various matrix elements, v_{ij}, for the general case are shown in Table 1.

2.34 The matrix product

$$C^{(k)} = C^{(1)} V^{k-1} \tag{2.48}$$

gives the distribution of probabilities for firing first shots at the k-th attacker. To obtain the penetration probability, Q_k, for the k-th attacker it is necessary to multiply the elements of $C^{(k)}$ by the probabilities of the requisite number of failures to permit a penetration. This can be accomplished by multiplying $C^{(k)}$ on the right by a column matrix E, whose elements are given by

$$e_j = q^{a-n} \tag{2.49}$$

TABLE 1

ELEMENTS OF THE TRANSFORMATION MATRIX

$n = 0, 1, 2, \ldots$		
j	i	v_{ij}
$j<(a-1)\tau - s + 1$	$i \le j + s - (n+1)\tau$	$\sum_{n=0}^{n \le (1+s-\tau-i)/\tau} pq^n$
$1<j \le (a-1)\tau - s + 1$	$i = j + s - (n+1)\tau$	pq^n
$(a-1)\tau - s + 1 < j \le a\tau - s + 1$	$i = j + s - (n+1)\tau$ $i = j + s$	q^n 1
All other $v_{ij} = 0$, and in particular if $j>a\tau - s + 1$, $v_{ij} = 0$ for all i . It should be noted that $i_{max} = j_{max} = a\tau$.		

where

$$1 + n\tau \leq j < 1 + (n+1)\tau \; (n=0,\ 1,\ 2,\ .\ .\ .\ ) \; . \qquad (2.50)$$

2.35 In principle a solution can now be obtained for an arbitrary constant spacing, but if it is necessary to look at the course of events in fine detail at small intervals of a shot, the dimensionality of the transformation matrix is correspondingly increased and it soon becomes necessary to use some means of computation more elaborate than a desk calculator. However, with an electronic computer there is no reason why the defense zone should not be broken down into as many sub-regions as necessary. If this is done, a set of transformation matrices depending only on $\underline{s}$ can be obtained. It would thus be possible to calculate what would happen if the attacker arrived with variable spacings by successively multiplying by the transformation matrices of proper $\underline{s}$. Further modifications would have to be made to take into account a variable spacing between shots.

PENETRATION PROBABILITY CURVES

2.36 The matrix method described has been programmed for an IBM 704 computer at the General Motors Research Center. Solutions have been obtained for $\underline{a}$ as high as 30 shots and for s/τ ranging from 0.1 to 0.9 shots.

2.37 Range of Parameters Considered. Penetration probabilities have obtained for all the following values of the parameters:

$s = 1,3(3)15$ [1/]

$s = 1$ [2/]

$\tau = 2(1)10$

$p = 0.80,\ 0.85,\ 0.90$

[1/] The number in parentheses is the interval between successive values of the parameters; e.g., 3(3)15 denotes the values 3,6,9,12,15.

[2/] It should be recalled that the quantity affecting the results of the computations is the dimensionless ratio s/τ .

The value of k, the number of attackers, is taken up to the value that results in a value of $P(k)$ equal to one or up to 30, whichever value is least. Table 2 shows a sample of the printed output of calculations of the $P(k)$, the probability that at least one of k attackers will penetrate the defense, for the choice of parameters $a=9$, $s=1$, $\tau=3$, and $p=0.85$.

2.38 Computations were also made for the two cases of a defense with greater depth, with the following values of the parameters:

$a = 20$

$\tau = 10$

$s = 1(1)9$

$p = 0.85$

and

$a = 30$

$\tau = 10$

$s = 1(1)6$

$p = 0.85$

The value of k was permitted to go as high as necessary to insure penetration.

2.39 Typical results of the calculations are shown in Figure 2, which shows the probabilities, $P(k)$, that at least one of k attackers will penetrate the defense for the case $a=9$, $p=0.85$, and for s/τ ranging from zero to one. Figure 3 shows some results for spacings with s/τ greater than unity. The cases calculated were for $a=3$ and $s/\tau = \frac{1}{2}$, 1, 3/2, and 2. The case for $s/\tau=0$, derived from a binomial distribution, was also calculated, as was the case of repeated independent arrivals ($s/\tau > 3$). The single-shot kill probability used was 0.85. The curves show that the expected number of attackers required is a very sensitive function of s/τ.

SIMPLE CASES

2.40 A consideration of several limiting cases is helpful in understanding the results of the calculations.

TABLE 2

SAMPLE RESULT OF PENETRATION-PROBABILITY CALCULATION

INPUT DATA

a	τ	s	p	k
9	3	1	0.85000	30

OUTPUT

k	P(k)
2	0.00000152
3	0.00004770
4	0.00014276
5	0.00152554
6	0.01138432
7	0.01879442
8	0.08693384
9	0.28889289
10	0.34305939
11	0.70867319
12	0.95856106
13	0.96477690
14	0.99999999
15	1.00000000

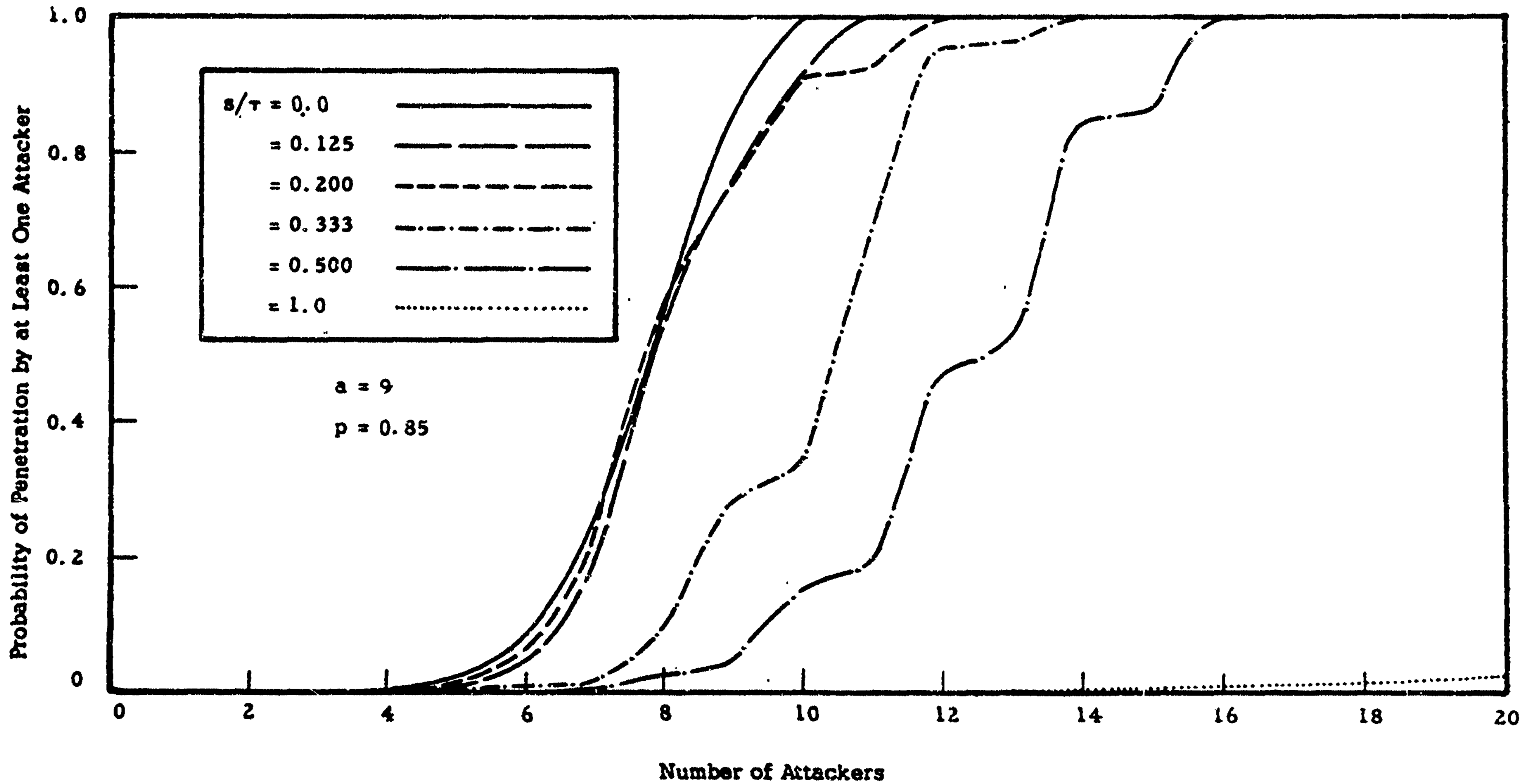

FIGURE 2. PENETRATION PROBABILITIES FOR FRACTIONAL SPACINGS

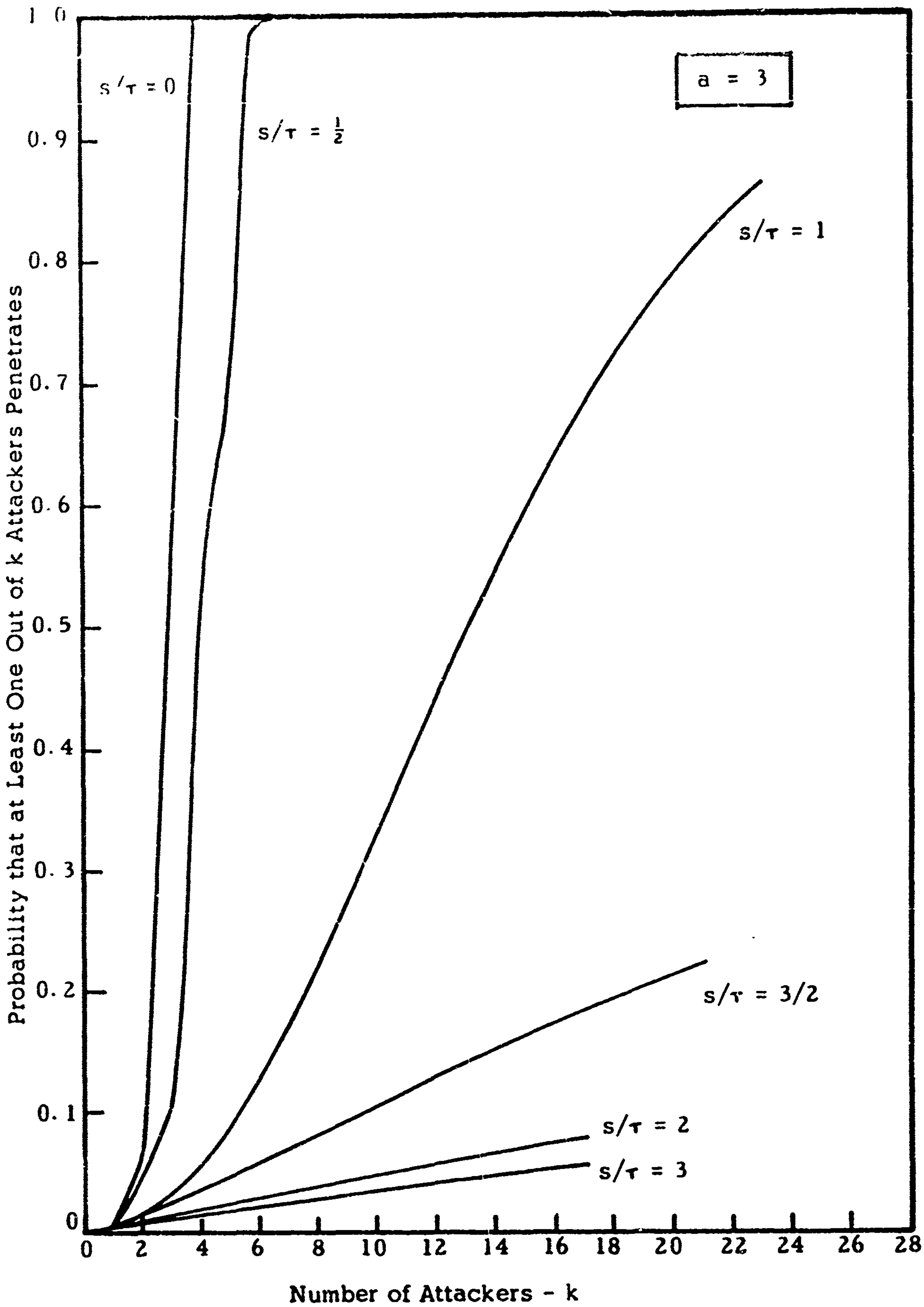

FIGURE 3. PENETRATION PROBABILITY CURVES

2.41 Simultaneous Arrivals. The first case discussed is that of simultaneous arrivals of attackers. It can be observed that if there is time for $\underline{a}$ shots at a single attacker, the defense can be considered to be allocated among $\underline{a}$ concentric zones surrounding the defended center. The attackers advance one zone at each shot fired by the defense no matter whether there is a success or a failure on the part of the defenders. Therefore, the probability of penetration is the probability of having at least k successes out of $\underline{a}$ trials, where each success occurs with a probability p. This probability is easily obtained by summing a binomial distribution for $\underline{a}$ trials with a probability of success p for individual trials and the required number of successes equal to or greater than k. In particular, for k greater than $(a+1)$ there will certainly be at least one penetration of the defenses. The mean number of attackers shot down is

$$\bar{k} = ap \tag{2.51}$$

so that an attack in strength ap is sufficient to achieve a penetration half of the time.

2.42 One-Shot Spacing. Another case is that of attackers arriving at a spacing equal to the interval between successive firings. For this case, intercepts will occur in the outermost defensive zone until there is a failure, after which the defense is forced to withdraw one zone. Therefore, penetration of the defenses occurs after $\underline{a}$ failures. The probability of having k successes before the occurrence of $\underline{a}$ failures is given by the so-called negative binomial distribution, and the mean number of successes is found to be

$$k = \frac{ap}{q} . \tag{2.52}$$

The variance about this mean is

$$\sigma^2 = \frac{ap}{q^2} . \tag{2.53}$$

2.43 Saturation Attack. It is more difficult to discuss the solutions given by the model for fractional spacings between zero and one, but some observations can be made on the maximum number of attackers required to saturate the defenses, and on the shape of the penetration probability curves. For each failure, the attackers are able to move in one zone closer, but even for a success, all the unopposed attackers move in $(1 - s/\tau)$ zones so that the maximum number of attackers required is

$$k_s = \frac{a}{1 - s/\tau} + 1 \; . \tag{2.54}$$

Also, on the average, an attacker will advance one zone unopposed every $1/(1 - s/\tau)$ shots, so that the depth of defense is decreased, and the penetration probability will display a sudden increase every $1/(1 - s/\tau)$ shots. The curves of Figure 2 display these sudden increases.

III. AVERAGE ATTRITION OF ATTACK FORCE

3.1 The calculations based on the model developed in Section II give the probability of penetration of at least one of k attackers as a function of the number of attackers, the spacing of the attackers, the depth of the defense, and the single-shot kill probability of the defensive missiles. The exact calculations of the model can be used to check estimates of the mean number of attackers required for penetration of the defenses.

3.2 An approximate estimate of the average attrition of an attack force is obtained by equating the mean engagement time $(a\tau+\overline{k}s)$ to the time $(\bar{n}\tau)$ that is required for the mean number of firings $\bar{n}$ that would achieve $\overline{k}$ kills. The approximation is:

$$\bar{n} = \overline{k}/p \tag{3.1}$$

$$a\tau + \overline{k}s = \bar{n}\tau \tag{3.2}$$

or

$$\overline{k} = \frac{ap}{1-(s/\tau)p} \ . \tag{3.3}$$

3.3 The quantity $\overline{k}$ is shown plotted against the calculated values of the mean number of attackers required to ensure penetration in Figure 4. Although there is some scatter of the calculated mean number of attackers around the line in Figure 4, the agreement between the calculated number and the estimated number is surprisingly good, and the expression given above for $\overline{k}$ serves as a reasonable estimate.

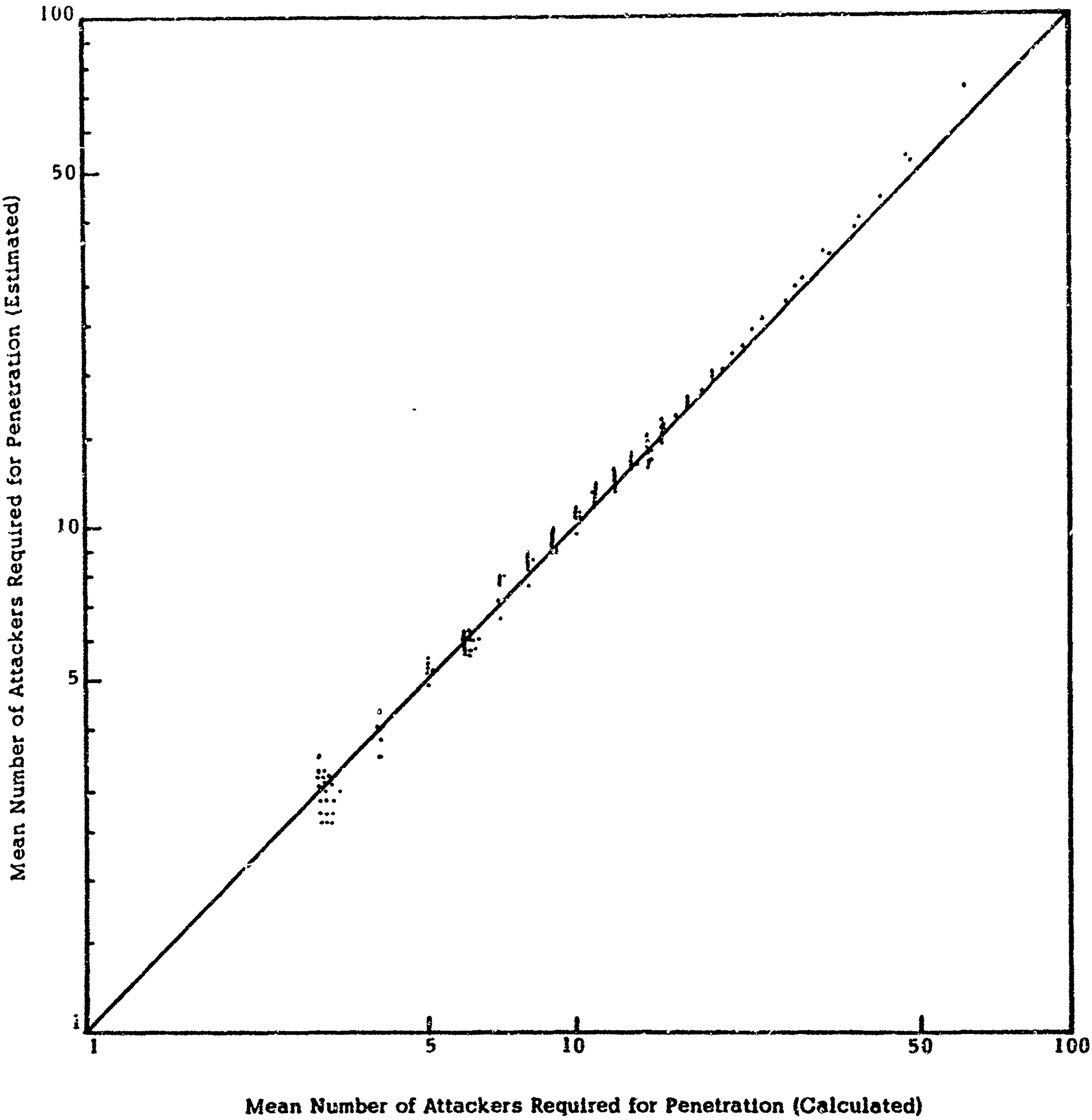

FIGURE 4. CORRELATION BETWEEN CALCULATED AND ESTIMATED NUMBER OF ATTACKERS REQUIRED FOR PENETRATION

3.4 Range of Validity of the Approximation. The approximation of paragraph (3.2) obviously fails when ps is equal to τ, for then the approximated $\bar{k}$ becomes infinite. However, when $s/\tau = 0$, Equation (3.3) reduces to

$$\bar{k} = ap \ , \tag{3.4}$$

which is the mean for simultaneous arrivals given in Equation (2.51). When $s = \tau$, Equation (3.3) reduces to

$$\bar{k} = ap/q \ , \tag{3.5}$$

which is the mean for a one-shot spacing given in Equation (2.52). Therefore, the approximation is valid as long as

$$s/\tau < 1 \ . \tag{3.6}$$

3.5 Effect of Spacing of Attackers. The expression for $\bar{k}$ is directly proportional to $\underline{a}$, the parameter representing the number of shots that could be made at a single attacker. To obtain $\bar{k}$, the quantity $\underline{a}$ is multiplied by a factor $p/(1 - ps/\tau)$, which gives a measure of the effectiveness of the defense as a function of p and s/τ. The quantity $p/(1 - ps/\tau)$ is plotted as a function of s/τ for various values of p in Figure 5. The curves show that if the attackers come in even 0.1 of a shot apart, about 10% more are needed than are needed for simultaneous arrivals, and the number required increases rapidly with increasing spacing.

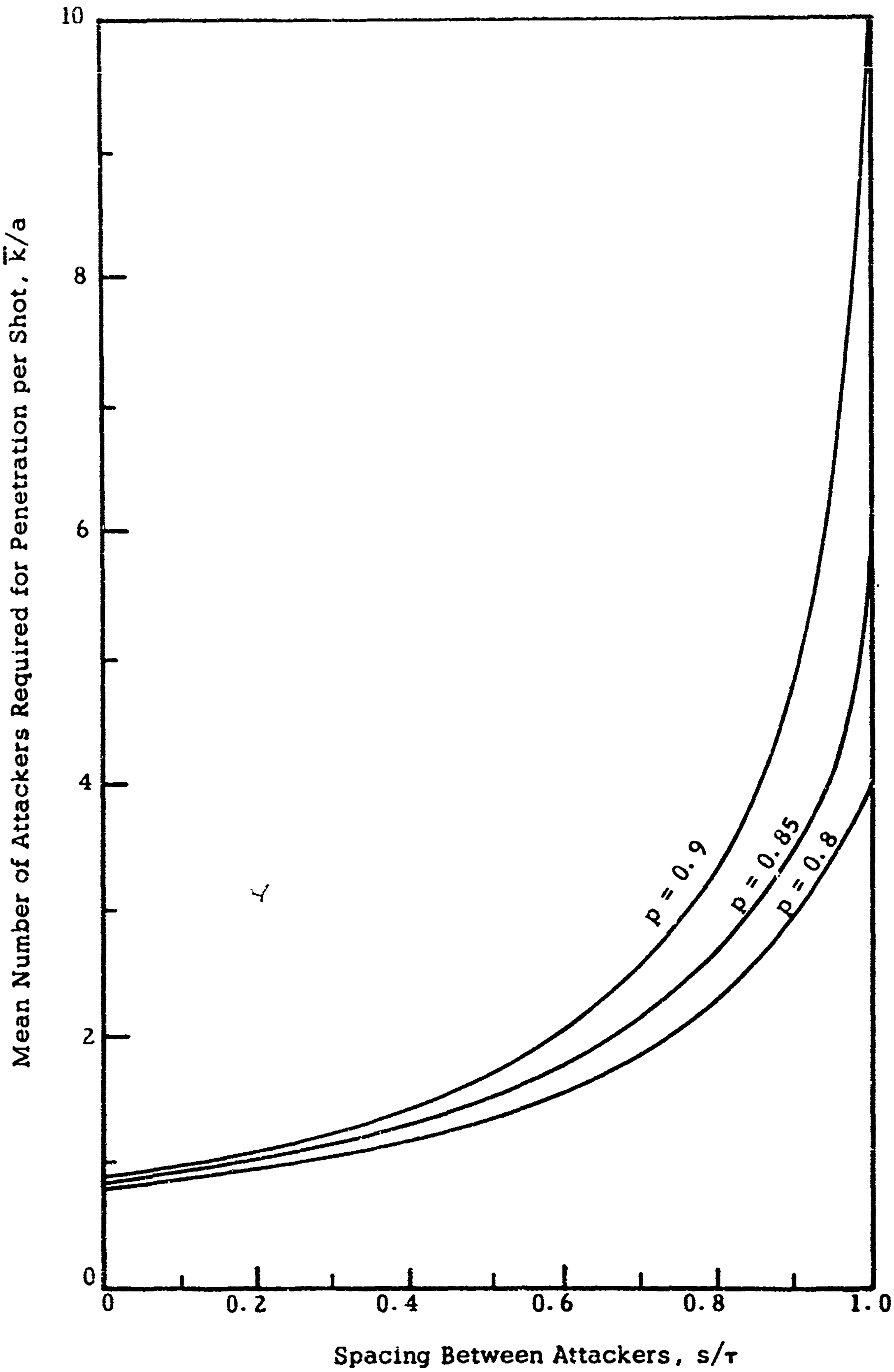

FIGURE 5. EFFECT OF SPACING ON PENETRATION PROBABILITY